Birationally rigid Fano fibre spaces. II

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In this paper we prove birational rigidity of large classes of Fano-Mori fibre spaces over a base of arbitrary dimension, bounded from above by a constant that depends on the dimension of the fibre only. In order to do that, we first show that if every fibre of a Fano-Mori fibre space satisfies certain natural conditions, then every birational map onto another Fano-Mori fibre space is fibre-wise. After that we construct large classes of fibre spaces (into Fano double spaces of index one and into Fano hypersurfaces of index one) which satisfy those conditions.

Bibliography: 35 titles.

Introduction

0.1. Birationally rigid Fano-Mori fibre spaces. In this paper we investigate the problem of birational rigidity of Fano-Mori fibre spaces $\pi: V \to S$. We assume that the base S is non-singular, the variety V has at most factorial terminal singularities, the anticanonical class $(-K_V)$ is relatively ample and

$$\operatorname{Pic} V = \mathbb{Z} K_V \oplus \pi^* \operatorname{Pic} S.$$

Let $\pi': V' \to S'$ be an arbitrary rationally connected fibre space, that is, a morphism of projective algebraic varieties, where the base S' and the fibre of general position $\pi'^{-1}(s')$, $s' \in S'$, are rationally connected and $\dim V = \dim V'$. Consider a birational map $\chi: V \dashrightarrow V'$ (provided they exist). In order to describe the properties of the map χ, of crucial importance is whether χ is fibre-wise or not, that is, whether this map transforms the fibres of the projection π into the fibres of the projection π'. It is expected (and confirmed by all known examples, see subsection 0.6), that the answer is positive if the fibre space π is "sufficiently twisted over the base". Investigating this problem, one can choose various classes of Fano-Mori fibre space and various interpretations of the property to be "twisted over the base". In the present paper we prove the following fact.

Theorem 1. *Assume that the Fano-Mori fibre space $\pi: V \to S$ satisfies the conditions*

(i) *every fibre $F_s = \pi^{-1}(s)$, $s \in S$, is a factorial Fano variety with terminal singularities and the Picard group* $\operatorname{Pic} F_s = \mathbb{Z} K_{F_s}$,

(ii) *for every effective divisor* $D \in |-nK_{F_s}|$ *on an arbitrary fibre* F_s *the pair* $(F_s, \frac{1}{n}D)$ *is log canonical, and for every mobile linear system* $\Sigma_s \subset |-nK_{F_s}|$ *the pair* $(F_s, \frac{1}{n}D)$ *is canonical for a general divisor* $D \in \Sigma_s$,

(iii) *for every mobile family* $\overline{\mathcal{C}}$ *of curves on the base* S*, sweeping out* S*, and a curve* $\overline{C} \in \overline{\mathcal{C}}$ *the class of the following algebraic cycle of dimension* $\dim F$ *for any positive* $N \geqslant 1$

$$-N(K_V \circ \pi^{-1}(\overline{C})) - F$$

(where F *is the fibre of the projection* π*) is not effective, that is, it is not rationally equivalent to an effective cycle of dimension* $\dim F$.

Then every birational map $\chi: V \dashrightarrow V'$ *onto the total space of a rationally connected fibre space* V'/S' *is fibre-wise, that is, there exists a rational dominant map* $\beta: S \dashrightarrow S'$*, such that the following diagram commutes*

$$\begin{array}{ccccc} & V & \stackrel{\chi}{\dashrightarrow} & V' & \\ \pi & \downarrow & & \downarrow & \pi' \\ & S & \stackrel{\beta}{\dashrightarrow} & S'. & \end{array}$$

Now we list the standard implications of Theorem 1, after which we discuss the point of how restrictive the conditions (i)-(iii) are.

Corollary 1. *In the assumptions of Theorem 1 on the variety* V *there are no structures of a rationally connected fibre space over a base of dimension higher than* $\dim S$*. In particular, the variety* V *is non-rational. Every birational self-map of the variety* V *is fibre-wise and induces a birational self-map of the base* S*, so that there is a natural homomorphism of groups* $\rho: \mathrm{Bir}\, V \to \mathrm{Bir}\, S$*, the kernel of which* $\mathrm{Ker}\, \rho$ *is the group* $\mathrm{Bir}\, F_\eta = \mathrm{Bir}(V/S)$ *of birational self-maps of the generic fibre* F_η *(over the generic non-closed point* η *of the base* S*), whereas the group* $\mathrm{Bir}\, V$ *is an extension of the normal subgroup* $\mathrm{Bir}\, F_\eta$ *by the group* $\Gamma = \rho(\mathrm{Bir}\, V) \subset \mathrm{Bir}\, S$*:*

$$1 \to \mathrm{Bir}\, F_\eta \to \mathrm{Bir}\, V \to \Gamma \to 1.$$

How restrictive are the conditions (i)-(iii)? The condition (iii) belongs to the same class of conditions as the well known K^2-condition and the K-condition for fibrations over $\mathbb{P}^1$ (see, for instance, [31, Chapter 4]) and the Sarkisov condition for conic bundles (see [32, 33]). This condition measures the "degree of twistedness" of the fibre space V/S over the base S. Below we illustrate this meaning of the condition (iii) by particular examples. We will see that this condition is not too restrictive: for a fixed method of constructing the fibre space V/S and a fixed "ambient" fibre space X/S the condition (iii) is satisfied by "almost all" families of fibre spaces V/S.

In terms of numerical geometry of the varieties V and S the condition (iii) can be expressed in the following way. Let

$$A^*(V) = \bigoplus_{i=0}^{\dim V} A^i(V)$$

be the numerical Chow ring of the variety V, graded by codimension. Set

$$A_i(V) = A^{\dim V - i}(V) \otimes \mathbb{R}$$

and denote by the symbol $A_i^{\mathrm{mov}}(V)$ the closed cone in $A_i(V)$, generated by the classes of mobile cycles, the families of which sweep out V, and by the symbol $A_i^+(V)$ the pseudoeffective cone in $A_i(V)$, generated by the classes of effective cycles. Furthermore, by the symbol $A_{i,\leqslant j}(V)$ we denote the linear subspace in $A_i(V)$, generated by the classes of subvarieties of dimension i, the image of which on S has dimension at most j. In the real space $A_{i,\leqslant j}(V)$ consider the closed cones $A_{i,\leqslant j}^{\mathrm{mov}}(V)$ of mobile and $A_{i,\leqslant j}^+(V)$ of pseudoeffective classes. In a similar way we define the real vector space $A_i(S)$ and the closed cones $A_i^{\mathrm{mov}}(S)$ $A_i^+(S)$. If $\delta = \dim F$ is the dimension of the fibre of the projection π, then the operation of taking the preimage generates a linear map

$$\pi^* A_i(S) \to A_{\delta+i,\leqslant i}(V),$$

whereas $\pi^*(A_i^+(S)) \subset A_{\delta+i,\leqslant i}^+(V)$ and $\pi^*(A_i^{\mathrm{mov}}(S)) \subset A_{\delta+i,\leqslant i}^{\mathrm{mov}}(V)$. Now let us consider the linear map

$$\gamma \colon A_1(S) \to A_{\delta,\leqslant 1}(V),$$

defined by the formula

$$z \mapsto -(K_V \cdot \pi^* z).$$

The condition (iii) means that the image of the cone $\gamma(A_1^{\mathrm{mov}}(S))$ is contained in the boundary of the pseudoeffective cone $A_{\delta,\leqslant 1}^+(V)$, that is,

$$\gamma(A_1^{\mathrm{mov}}(S)) \cap \mathrm{Int}\, A_{\delta,\leqslant 1}^+(V) = \emptyset.$$

More precisely, for any class $z \in A_1^{\mathrm{mov}}(S)$ the intersection of the closed ray

$$\{\gamma(z) - t[F] \,|\, t \in \mathbb{R}_+\}$$

(where $[F] \in \mathrm{Int}\, A_{\delta,\leqslant 1}^+(V)$ is the class of the fibre of the projection π) with the cone $A_{\delta,\leqslant 1}^+(V)$ either is empty or consists of just one point $\gamma(z)$.

One may suggest that the condition (iii) is close to a precise one ("if and only if"), that is, its violation (or an essential deviation from this condition) implies the existence of another structure of a Fano-Mori fibre space on the variety V.

The following remark gives an obvious way to check the condition (iii).

Remark 0.1. Assume that on the variety V there is a numerically effective divisorial class L such that $(L^\delta \cdot F) > 0$ and the linear function $(\cdot L^\delta)$ is non-positive on the cone $\gamma(A_1^{\mathrm{mov}}(S))$, that is to say, for any mobile curve $\overline{C}$ on S the inequality

$$\left(L^\delta \cdot K_V \cdot \pi^{-1}(\overline{C})\right) \geqslant 0 \qquad (1)$$

holds. Then the condition (iii) is obviously satisfied.

The conditions (i) and (ii), however, are much more restrictive. They mean that *all* fibres of the projection π are varieties of sufficiently general position in their family. This implies that the dimension of the base for a fixed family of fibres is bounded from above (by a constant depending on the particular family, to which the fibres belong). In the examples considered in the present paper, for a sufficiently high dimension of the fibre $\delta = \dim F$ the dimension of the base is bounded from above by a number of order $\frac{1}{2}\delta^2$. Recall that up to now not a single example was known of a fibration into higher dimensional Fano varieties over a base of dimension two and higher with just one structure of a rationally connected fibre space (for a brief historical survey, see subsection 0.5).

0.2. Fibrations into double spaces of index one. By the symbol $\mathbb{P}$ we denote the projective space $\mathbb{P}^M$, $M \geqslant 5$. Let $\mathcal{W} = \mathbb{P}(H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(2M)))$ be the space of hypersurfaces of degree $2M$ in $\mathbb{P}$. The following fact is true.

Theorem 2. *There exists a Zariski open subset $\mathcal{W}_{\rm reg} \subset \mathcal{W}$, such that for any hypersurface $W \in \mathcal{W}_{\rm reg}$ the double cover $\sigma\colon F \to \mathbb{P}$, branched over W, satisfies the conditions (i) and (ii) of Theorem 1, and moreover, the estimate*

$$\operatorname{codim}((\mathcal{W} \setminus \mathcal{W}_{\rm reg}) \subset \mathcal{W}) \geqslant \frac{(M-4)(M-1)}{2}$$

holds.

An explicit description of the set $\mathcal{W}_{\rm reg}$ and a proof of Theorem 2 are given in §2. Fix a number $M \geqslant 5$ and a non-singular rationally connected variety S of dimension $\dim S < \frac{1}{2}(M-4)(M-1)$. Let $\mathcal{L}$ be a locally free sheaf of rank $M+1$ on S and $X = \mathbb{P}(\mathcal{L}) = \mathbf{Proj} \mathop{\oplus}\limits_{i=0}^{\infty} \mathcal{L}^{\otimes i}$ the corresponding $\mathbb{P}^M$-bundle. We may assume that $\mathcal{L}$ is generated by its sections, so that the sheaf $\mathcal{O}_{\mathbb{P}(\mathcal{L})}(1)$ is also generated by the sections. Let $L \in \operatorname{Pic} X$ be the class of that sheaf, so that

$$\operatorname{Pic} X = \mathbb{Z}L \oplus \pi_X^* \operatorname{Pic} S,$$

where $\pi_X\colon X \to S$ is the natural projection. Take a general divisor $U \in |2(ML + \pi_X^* R)|$, where $R \in \operatorname{Pic} S$ is some class. If this system is sufficiently mobile, then by the assumption about the dimension of the base S and Theorem 2 we may assume that for every point $s \in S$ the hypersurface $U_s = U \cap \pi_X^{-1}(s) \in \mathcal{W}_{\rm reg}$, and for that reason the double space branched over U_s, satisfies the conditions (i) and (ii) of Theorem 1. Let $\sigma\colon V \to X$ the double cover branched over U. Set $\pi = \pi_X \circ \sigma\colon V \to S$, so that V is a fibration into Fano double spaces of index one over S. Recall that the divisor $U \in |2(ML + \pi_X^* R)|$ is assumed to be sufficiently general.

Theorem 3. *Assume that the divisorial class $(K_S + R)$ is pseudoeffective. Then for the fibre space $\pi\colon V \to S$ the claims of Theorem 1 and Corollary 1 hold. In particular,*

$$\operatorname{Bir} V = \operatorname{Aut} V = \mathbb{Z}/2\mathbb{Z}$$

is the cyclic group of order 2.

Proof. Since the conditions (i) and (ii) of Theorem 1 are satisfied by construction of the variety V, it remains to check the condition (iii). Let us use Remark 0.1.

Elementary computations show that the inequality (1) up to a positive factor is the inequality

$$((K_S + R) \cdot \overline{C}) \geqslant 0.$$

Since the curve $\overline{C}$ belongs to a mobile family, sweeping out the base S, the last inequality holds if the class $(K_S + R)$ is pseudoeffective. Q.E.D. for the theorem.

Example 0.2. Take $S = \mathbb{P}^m$, where $m < \frac{1}{2}(M-4)(M-1)$, $X = \mathbb{P}^M \times \mathbb{P}^m$ and W_X is a generic hypersurface of bidegree $(2M, 2l)$, where $l \geqslant m+1$. Then for the double cover $\sigma\colon V \to X$, branched over W_X, the claims of Theorem 1 and Corollary 1 are true. Note that for $l \leqslant m$ on the double cover V there is another structure of a Fano fibre space: it is given by the projection $\pi_1\colon V \to \mathbb{P}^M$. Therefore, the condition (iii) of Theorem 1 and its realization in Theorem 3 turn out to be precise.

0.3. Fibrations into Fano hypersurfaces of index one. The symbol $\mathbb{P}$ still stands for the projective space $\mathbb{P}^M$, $M \geqslant 10$. Fix M. Let $\mathcal{F} = \mathbb{P}(H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(M)))$ be the space of hypersurfaces of degree M in $\mathbb{P}$. The following fact is true.

Theorem 4. *There is a Zariski open subset $\mathcal{F}_{\rm reg} \subset \mathcal{F}$, such that every hypersurface $F \in \mathcal{F}_{\rm reg}$ satisfies the conditions (i) and (ii) of Theorem 1, and the following estimate holds:*

$$\operatorname{codim}((\mathcal{F} \setminus \mathcal{F}_{\rm reg}) \subset \mathcal{F}) \geqslant \frac{(M-7)(M-6)}{2} - 5. \qquad (2)$$

An explicit description of the subset $\mathcal{F}_{\rm reg}$ and a proof of Theorem 4 are given in §2-3. Fix a non-singular rationally connected variety S of dimension $\dim S < \frac{1}{2}(M-7)(M-6) - 5$. As in subsection 0.2, let $\mathcal{L}$ be a locally free sheaf of rank $M+1$ on S and $X = \mathbb{P}(\mathcal{L}) = \mathbf{Proj} \mathop{\oplus}\limits_{i=0}^{\infty} \mathcal{L}^{\otimes i}$ the corresponding $\mathbb{P}^M$-bundle in the sense of Grothendieck; we assume that $\mathcal{L}$ is generated by global sections. Let $\pi_X\colon X \to S$ be the projection, $L \in \operatorname{Pic} X$ the class of the sheaf $\mathcal{O}_{\mathbb{P}(\mathcal{L})}(1)$. Consider a general divisor

$$V \in |ML + \pi_X^* R|,$$

where $R \in \operatorname{Pic} S$ is some divisor on the base. By the assumption about the dimension of the base made above and Theorem 4 we may assume that the Fano fibre space $\pi\colon V \to S$, where $\pi = \pi_X|_V$, satisfies the conditions (i) and (ii) of Theorem 1.

Theorem 5. *Assume that the divisorial class $(K_S + \left(1 - \frac{1}{M}\right) R)$ is pseudoeffective. Then for the Fano fibre space $\pi\colon V \to S$ the claims of Theorem 1 and Corollary are true. In particular, the group*

$$\operatorname{Bir} V = \operatorname{Aut} V$$

is trivial.

Proof. The conditions (i) and (ii) of Theorem 1 are satisfied by the generality of the divisor V. The inequality (1) up to a positive factor is the same as the inequality

$$((MK_S + (M-1)R) \cdot \overline{C}) \geqslant 0.$$

Therefore, by Remark 0.1, the condition (iii) of Theorem 1 also holds. Q.E.D. for the theorem.

Example 0.2. Take $S = \mathbb{P}^m$, where $m \leqslant \frac{1}{2}(M-7)(M-6)-6$, $X = \mathbb{P}^M \times \mathbb{P}^m$ and $V \subset X$ is a sufficiently general hypersurface of bidegree (M, l), where l satisfies the inequality

$$l \geqslant \frac{M}{M-1}(m+1).$$

Then the Fano fibre space $V/\mathbb{P}^m$ satisfies all assumptions of Theorem 1 and therefore for this fibre space the claim of Theorem 1 and that of Corollary 1 are true. Note that for $l \leqslant m$ on the variety V there is another structure of a Fano fibre space, given by the projection $V \to \mathbb{P}^M$. Note also that if we fix the dimension m of the base, then for $M \geqslant m$ the condition of Theorem 5 is close to the optimal one: it is satisfied for $l \geqslant m+2$, so that the only value of the integral parameter l, for which the problem of birational rigidity of the fibre space $V/\mathbb{P}^m$ remains open, is $l = m+1$. In that case the projection $V \to \mathbb{P}^M$ is a K-trivial fibre space.

0.4. The structure of the paper. The present paper is organized in the following way. In §1 we prove Theorem 1. After that, in §2 we deal with the conditions of general position, which should be satisfied for every fibre of the fibre space V/S in order for the conditions (i) and (ii) of Theorem 1 to hold. The conditions of general position (regularity) are given for Fano double spaces of index one and Fano hypersurfaces of index one. This makes it possible to define the sets $\mathcal{W}_{\rm reg}$ and $\mathcal{F}_{\rm reg}$ and prove Theorem 2 and carry out the preparational work for the proof of Theorem 4, the main technical fact of the present paper, which implies Theorem 5, geometrically the most impressive result of this paper, in an obvious way.

In §3 we complete the proof of Theorem 4, more precisely we show that the condition (ii) of Theorem 1 is satisfied for a regular Fano hypersurface $F \in \mathcal{F}_{\rm reg}$. The proof makes use a combination of the technique of hypertangent divisors and the inversion of adjunction. Note that the approach of the present paper corresponds to the linear method of proving birational rigidity, see [31, Chapter 7]; the technique of the quadratic method (in the first place, the technique of counting multiplicities) is not used.

The assumption in Theorem 1 that the base S of the fibre space V/S is non-singular seems to be unnecessary and could be replaced by the condition that the singularities are at most terminal and ($\mathbb{Q}$-)factorial.

0.5. Historical remarks and acknowledgements. The starting point of studying birational geometry of rationally connected fibre spaces seems to be the use of de Jonqiere transformations (see, for instance, [11]). In the modern algebraic geometry the objects of this type started to be systematically investigated in the works of V.A.Iskovskikh and M.Kh.Gizatullin about pencils of rational curves [12, 13, 6] over non-closed fields, which followed the investigation of the "absolute" case in the papers of Yu.I.Manin [16, 17, 18]. We also point out the paper of I.V.Dolgachev [4], which started (in the modern period) the study of K-trivial fibrations.

After the breakthrough in three-dimensional birational geometry that was made in the classical paper of V.A.Iskovskikh and Yu.I.Manin on the three-dimensional quartic [14] the problems of the "relative" three-dimensional birational geometry were the next to be investigated, that is, the task was to describe birational maps of three-dimensional algebraic varieties, fibred into conics over a rational surface or into del Pezzo surfaces over $\mathbb{P}^1$. The famous Sarkisov theorem gave an almost complete solution of the question of birational rigidity for conic bundles [32, 33]. A similar question for the pencils of del Pezzo surfaces remained absolutely open until 1996 [19]; see the introduction to the last paper about the reasons of those difficulties (the test class construction turned out to be unsuitable for studying the varieties of that type).

The method of proving birational rigidity, realized in [19], generalized well into the arbitrary dimension, for varieties fibred into Fano varieties over $\mathbb{P}^1$. In a long series of papers [21, 34, 35, 23, 24, 26, 27, 29] birational rigidity was shown for many classes of Fano fibre space over $\mathbb{P}^1$. At the same time, the birational geometry of the remaining families of three-dimensional varieties with a pencil of del Pezzo surfaces of degree 1 and 2 was investigated [7, 8, 9, 10]; in that direction the results that were obtained were nearly exhaustive. However, the base of the fibre spaces under investigation remained one-dimensional and even Fano fibrations over surfaces seemed to be out of reach.

The only exception in that series of results was the theorem about Fano direct products [25] and the papers about direct products that followed [28, 2]. In those papers the Fano fibre spaces under consideration had the both the base and the fibre of arbitrary dimension. However, the fibre spaces themselves were very special (direct products) and could not pretend to be *typical* Fano fibre spaces.

The present paper gives, at long last, numerous examples of typical birationally rigid Fano fibre spaces with the base and fibre of high dimension (for a fixed dimension of the fibre δ the dimension of the base is bounded by a constant of order $\sim \frac{1}{2}\delta^2$). Theorem 1 can be viewed as a realization of the well known principle: the "sufficient twistedness" of a fibre space over the base implies birational rigidity. This principle was many times confirmed in the class of fibrations over $\mathbb{P}^1$; now it is extended to the the class of fibre spaces over a base of arbitrary dimension.

The main object of study in this paper is a fibre space into Fano hypersurfaces of index one, so that it is a follow up of the paper [21]. From the technical viewpoint, the predecessors of this paper are [28, 30], where the *linear* method of proving birational rigidity was developed. It is possible, however, that the quadratic techniques could be applied to the class of Fano fibre spaces over a base of arbitrary dimension as well.

Various technical moments related to the arguments of the present paper were discussed by the author in his talks given in 2009-2014 at Steklov Institute of Mathematics. The author is grateful to the members of the divisions of Algebraic Geometry and Algebra and Number Theory for the interest in his work. The author also thanks his colleagues in the Algebraic Geometry research group at the University of Liverpool for the creative atmosphere and general support.

1 Birationally rigid fibre spaces

In this section we prove Theorem 1. We do it in three steps: first, assuming that the birational map $\chi: V \dashrightarrow V'$ is not fibre-wise, we prove the existence of a maximal singularity of the map χ, covering the base S' (subsection 1.1). After that, we construct such a sequence of blow ups of the base $S^+ \to S$, that the image of every maximal singularity on S is a prime divisor (subsection 1.2). Finally, using a very mobile family of curves contracted by the projection π', we obtain a contradiction with the condition (iii) of Theorem 1 (subsection 1.3). This implies that the map $\chi: V \dashrightarrow V'$ is fibre-wise, which completes the proof of Theorem 1.

1.1. Maximal singularities of birational maps. In the notations of Theorem 1 fix a birational map $\chi: V \dashrightarrow V'$ onto the total space V' of a rationally connected fibre space $\pi': V' \to S'$. Consider any very ample linear system $\overline{\Sigma'}$ on S'. Let $\Sigma' = (\pi')^* \overline{\Sigma'}$ be its pull back on V', so that the divisors $D' \in \Sigma'$ are composed from the fibres of the projection π', and for that reason for any curve $C \subset V'$, contracted by the projection π' we have $(D' \cdot C) = 0$. The linear system Σ' is obviously mobile. Let

$$\Sigma = (\chi^{-1})_* \Sigma' \subset | - nK_V + \pi^* Y|$$

be its strict transform on V, where $n \in \mathbb{Z}_+$.

Lemma 1.1. *For any mobile family of curves $\overline{C} \in \overline{\mathcal{C}}$ on S, sweeping out S, the inequality $(\overline{C} \cdot Y) \geqslant 0$ holds, that is to say, the numerical class of the divisor Y is non-negative on the cone $A_1^{\mathrm{mov}}(S)$.*

Proof. This is almost obvious. For a general divisor $D \in \Sigma$ the cycle $(D \circ \pi^{-1}(\overline{C}))$ is effective. Its class is $-n(K_V \circ \pi^{-1}(\overline{C})) + (Y \cdot \overline{C})F$, so that by the condition (iii) the claim of the lemma follows. Q.E.D.

Obviously, the map χ is fibre-wise if and only if $n = 0$. Therefore, if $n = 0$, then the claim of Theorem 1 holds. So let us assume that $n \geq 1$ and show that this assumption leads to a contradiction.

The linear system Σ is mobile. Let us resolve the singularities of the map χ: let

$$\varphi: \widetilde{V} \to V$$

be a birational morphism (a composition of blow ups with non-singular centres), where $\widetilde{V}$ is non-singular and the composition $\chi \circ \varphi: \widetilde{V} \dashrightarrow V'$ is regular. Furthermore, consider the set $\mathcal{E}$ of prime divisors on $\widetilde{V}$, satisfying the following conditions:

- every divisor $E \in \mathcal{E}$ is φ-exceptional,
- for every $E \in \mathcal{E}$ the closed set $\chi \circ \varphi(E) \subset V'$ is a prime divisor on V',
- the set $\chi \circ \varphi(E)$ for every $E \in \mathcal{E}$ covers the base: $\pi'[\chi \circ \varphi(E)] = S'$.

Setting $\widetilde{K} = K_{\widetilde{V}}$, write down

$$\widetilde{\Sigma} \subset | - n\widetilde{K} + (\pi^* Y - \sum_{E \in \mathcal{E}} \varepsilon(E) E) + \Xi|,$$

where $\widetilde{\Sigma}$, as usual, is the strict transform of the mobile linear system Σ on $\widetilde{V}$, $\varepsilon(E) \in \mathbb{Z}$ is some coefficient and Ξ stands for a linear combination of φ-exceptional divisors which do not belong to the set $\mathcal{E}$.

Definition 1.1. An exceptional divisor $E \in \mathcal{E}$ is called *a maximal singularity* of the map χ, if $\varepsilon(E) > 0$.

Obviously, a maximal singularity satisfies *the Noether-Fano inequality*

$$\operatorname{ord}_E \varphi^* \Sigma > na(E),$$

where $a(E) = a(E, V)$ is the discrepancy of the divisor E with respect to V. In this paper we somewhat modify the standard concept of a maximal singularity, requiring in addition that it is realized by a divisor on V', covering the base. Let $\mathcal{M} \subset \mathcal{E}$ be the set of all maximal singularities.

Proposition 1.1. *Maximal singularities do exist:* $\mathcal{M} \neq \emptyset$.

Proof. Assume the converse, that is, for any $E \in \mathcal{E}$ the inequality $\varepsilon(E) \leq 0$ holds. Let $\mathcal{C}'$ be a family of rational curves on V', satisfying the following conditions:

- the curves $C' \in \mathcal{C}'$ are contracted by the projection π',
- the curves $C' \in \mathcal{C}'$ sweep out a dense open subset in V',
- the curves $C' \in \mathcal{C}'$ do not intersect the set of points where the rational map $(\chi \circ \varphi)^{-1} \colon V' \dashrightarrow \widetilde{V}$ is not well defined.

Apart from that, we assume that a general curve $C' \in \mathcal{C}'$ intersects every divisor $\chi \circ \varphi(E)$, $E \in \mathcal{E}$, transversally at points of general position. Such a family of curves we will call *very mobile*. Obviously, very mobile families of rational curves do exist.

Let $\widetilde{C} \cong C'$ be the inverse image of the curve $C' \in \mathcal{C}'$ on $\widetilde{V}$. Since the linear system Σ' is pulled back from the base, for a divisor $\widetilde{D} \in \widetilde{\Sigma}$ we have the equality $(\widetilde{C} \cdot \widetilde{D}) = 0$. On the other hand, $(\widetilde{C} \cdot \widetilde{K}) = (C' \cdot K_{V'}) < 0$ and

$$(\widetilde{C} \cdot (\pi^* Y - \sum_{E \in \mathcal{E}} \varepsilon(E) E)) \geqslant 0,$$

since by the condition (iii) of our theorem $(\widetilde{C} \cdot \pi^* Y) \geqslant 0$ and by assumption $-\varepsilon(E) \in \mathbb{Z}_+$ for all $E \in \mathcal{E}$. Finally, the divisor Ξ (which is not necessarily effective) is a linear combination of such φ-exceptional divisors $R \subset \widetilde{V}$, that $\pi'[\chi \circ \varphi(R)]$ is a proper closed subset of the base S'. So we have the equality $(\widetilde{C} \cdot \Xi) = 0$. This implies that

$$(\widetilde{C} \cdot \widetilde{D}) \geqslant n > 0,$$

which is a contradiction. Therefore, $\mathcal{M} \neq \emptyset$. Q.E.D. for the proposition.

Proposition 1.2. *For any maximal singularity $E \subset \mathcal{M}$ its center*

$$\operatorname{centre}(E, V) = \varphi(E)$$

on V does not cover the base: $\pi(\mathrm{centre}(E,V)) \subset S$ is a proper closed subset of the variety S.

Proof. Assume the converse: the centre of some maximal singularity $E \in \mathcal{M}$ covers the base: $\pi(\mathrm{centre}(E,V)) = S$. Let $F = \pi^{-1}(s)$, $s \in S$ be a fibre of general position. By assumption the strict transform $\widetilde{F}$ of the fibre F on $\widetilde{V}$ has a non-empty intersection with E, and for that reason every irreducible component of the intersection $\widetilde{F} \cap E$ is a maximal singularity of the mobile linear system $\Sigma_F = \Sigma|_F \subset |-nK_F|$. However, by the condition (ii) of Theorem 1 on the variety F there are no mobile linear systems with a maximal singularity. This contradiction proves the proposition.

1.2. The birational modification of the base of the fibre space V/S. Now let us construct a sequence of blow ups of the base, the composition of which is a birational morphism $\sigma_S\colon S^+ \to S$, and the corresponding sequence of blow ups of the variety V, the composition of which is a birational morphism $\sigma\colon V^+ \to V$, where $V^+ = V \times_S S^+$, so that the following diagram commutes

$$\begin{array}{ccccc} & V^+ & \stackrel{\sigma}{\to} & V & \\ \pi_+ & \downarrow & & \downarrow & \pi \\ & S^+ & \stackrel{\sigma_S}{\to} & S. & \end{array}$$

The birational morphism σ_S is constructed inductively as a composition of elementary blow ups $\overline{\sigma}_i\colon S_i \to S_{i-1}$, $i = 1, \ldots,$ where $S_0 = S$. Assume that $\overline{\sigma}_i$ are already constructed for $i \leqslant k$ (if $k = 0$, then we start with the base S). Set $V_k = V \times_S S_k$ and let $\pi_k\colon V_k \to S_k$ be the projection. Consider the irreducible closed subsets

$$\pi_k(\mathrm{centre}(E, V_k)) \subset S_k, \tag{3}$$

where E runs through the set $\mathcal{M}$. By Proposition 1.2, all these subsets are proper subsets of the base S_k. If all of them are prime divisors on S_k, we stop the procedure: set $S^+ = S_k$ and $V^+ = V_k$. Otherwise, for $\overline{\sigma}_{k+1}$ we take the blow up of any inclusion-minimal set (3) for all $E \in \mathcal{M}$.

It is easy to check that the sequence of blow ups $\overline{\sigma}$ terminates. Indeed, set

$$\alpha_k = \sum_{E \in \mathcal{M}} a(E, V_k).$$

Since the birational morphism $\sigma_k\colon V_k \to V_{k-1}$ is the blow up of a closed irreducible subset, containing the centre of one of the divisors $E \in \mathcal{M}$ on V_{k-1}, we get the inequality $\alpha_{k+1} < \alpha_k$. The numbers α_i are by construction non-negative, which implies that the sequence of blow ups $\overline{\sigma}_i$ is finite. Therefore, for any maximal singularity $E \in \mathcal{M}$ the closed subset $\pi_+(\mathrm{centre}(E, V^+)) \subset S^+$ is a prime divisor.

1.3. The mobile family of curves. Again let us consider a very mobile family of curves $\mathcal{C}'$ on V' and its strict transform $\mathcal{C}^+$ on V^+. Let $C^+ \in \mathcal{C}^+$ be a general curve and $\overline{C^+} = \pi_+(C^+)$ the corresponding curve of the family $\overline{\mathcal{C}^+}$ on S^+.

Furthermore, let Σ^+ be the strict transform of the linear system Σ on V^+. For some class of divisors Y^+ on S^+ we have:

$$\Sigma^+ \subset |-nK^+ + \pi_+^* Y^+|,$$

where for simplicity of notation $K^+ = K_{V^+}$. Note that even if Y is an effective or mobile class on S, in this case Y^+ is not its strict transform on S^+, that is to say, we violate the principle of notations. The following observation is crucial.

Proposition 1.3. *The inequality*

$$(\overline{C^+} \cdot Y^+) < 0$$

holds. In particular, the class Y^+ is not pseudoeffective.

Proof. Assume the converse:

$$(C^+ \cdot \pi^* Y^+) = (\overline{C^+} \cdot Y^+) \geqslant 0.$$

We may assume that the resolution of singularities φ of the map χ filters through the sequence of blow ups $\sigma\colon V^+ \to V$, so that for the strict transform $\widetilde{\Sigma}$ of the linear system Σ on $\widetilde{V}$ we have

$$\widetilde{\Sigma} \subset |-n\widetilde{K} + (\pi_+^* Y^+ - \sum_{E\in\mathcal{E}} \widetilde{\varepsilon}(E)E) + \widetilde{\Xi}|,$$

where $\widetilde{K} = K_{\widetilde{V}}$, $\widetilde{\varepsilon}(E) \in \mathbb{Z}$ and $\widetilde{\Xi}$ is a linear combination of exceptional divisors of the birational morphism $\widetilde{V} \to V^+$, which are not in the set $\mathcal{E}$. For the strict transform $\widetilde{C} \in \widetilde{\mathcal{C}}$ of the curve $C^+ \in \mathcal{C}^+$ and the divisor $\widetilde{D} \in \widetilde{\Sigma}$ we have, as in the proof of Proposition 1.1, the equality $(\widetilde{C} \cdot \widetilde{D}) = 0$. By the construction of the divisor $\widetilde{\Xi}$ we have $(\widetilde{C} \cdot \widetilde{\Xi}) = 0$. Finally, $(\widetilde{C} \cdot \widetilde{K}) < 0$, whence we conclude that

$$(\widetilde{C} \cdot (\pi_+^* Y^+ - \sum_{E\in\mathcal{E}} \widetilde{\varepsilon}(E)E) < 0.$$

By our assumption for at least one divisor $E \in \mathcal{E}$ we have the inequality $\widetilde{\varepsilon}(E) > 0$. This divisor is automatically a maximal singularity, $E \in \mathcal{M}$. By our construction, however, we can say more: E is a maximal singularity for the mobile linear system Σ^+ as well, that is, the pair $\left(V^+, \frac{1}{n}\Sigma^+\right)$ is not canonical and E realizes a non-canonical singularity of that pair.

However, $\pi_+(\mathrm{centre}(E, V^+)) = \overline{E} \subset S^+$ is a prime divisor, so that $\pi_+^{-1}(\overline{E}) \subset V^+$ is also a prime divisor. The linear system Σ^+ has no fixed components, therefore for a general point $s \in \overline{E}$ and the corresponding fibre $F = \pi_+^{-1}(s) \subset V^+$ we have: the linear system $\Sigma_F = \Sigma^+|_F \subset |-nK_F|$ is non-empty and for $D_F \in \Sigma_F$ the pair $\left(F, \frac{1}{n}D_F\right)$ is non log canonical by the inversion of adjunction (see [15]). This contradicts the condition (ii) of our theorem. Proposition 1.3 is shown. Q.E.D.

Finally, let us complete the proof of Theorem 1. Let us write down explicitly the divisor $\pi_+^* Y^+$ in terms of the partial resolution σ. Let $\mathcal{E}^+$ be the set of all exceptional divisors of the morphism σ, the image of which on V' is a divisor and covers the base S'. Therefore, $\mathcal{E}^+$ can be identified with a subset of the set $\mathcal{E}$. In the course of the proof of Proposition 1.3 we established that

$$\mathcal{M}^+ = \mathcal{M} \cap \mathcal{E}^+ \neq \emptyset.$$

Now we write

$$\pi_+^* Y^+ = \pi^* Y - \sum_{E \in \mathcal{E}^+} \varepsilon_+(E) E + \Xi^+.$$

Besides, we have

$$K^+ = \sigma^* K_V + \sum_{E \in \mathcal{E}^+} a_+(E) E + \Xi_K,$$

where all coefficients $a_+(E)$ are positive and the divisor Ξ_K is effective, pulled back from the base S^+ and the image of each of its irreducible component on V' has codimension at least 2, so that the general curve $C^+ \in \mathcal{C}^+$ does not intersect the support of the divisor Ξ_K. Let $C \in \mathcal{C}$ be its image on the original variety V and $\overline{C} = \pi(C) \in \overline{\mathcal{C}}$ the projection of the curve C on the base S. For a general divisor $D \in \Sigma$ and its strict transform $D^+ \in \Sigma^+$ on V^+ the scheme-theoretic intersection $(D^+ \circ \pi_+^{-1}(\overline{C^+}))$ is well defined, it is an effective cycle of dimension $\delta = \dim F$ on V^+. For its numerical class we have the presentation

$$\begin{aligned} (D^+ \circ \pi_+^{-1}(\overline{C^+})) \sim -n(\sigma^* K_V \circ \pi_+^{-1}(\overline{C^+})) + \\ + \left(\left[\sum_{E \in \mathcal{E}^+} (-n a_+(E) - \varepsilon_+(E)) E \right] \cdot C^+ \right) F. \end{aligned} \tag{4}$$

Since $(\overline{C} \cdot Y) \geqslant 0$ and $(C^+ \cdot \pi_+^* Y^+) < 0$, we have

$$\left(- \left[\sum_{E \in \mathcal{E}^+} \varepsilon_+(E) E \right] \cdot C^+ \right) < 0,$$

so that in the formula (4) the intersection of the divisor in square brackets with C^+ is negative. Therefore,

$$\sigma_*(D^+ \circ \pi_+^{-1}(\overline{C^+})) \sim -n(K_V \circ \pi^{-1}(\overline{C})) + bF,$$

where $b < 0$. Since on the left we have an effective cycle of dimension δ on V, we obtain a contradiction with the condition (iii) of our theorem. Proof of Theorem 1 is complete. Q.E.D.

2 Varieties of general position

In this section we state the explicit local conditions of general position for the double spaces (subsection 2.1) and hypersurfaces (subsection 2.2), defining the sets

$\mathcal{W}_{\rm reg} \subset \mathcal{W}$ and $\mathcal{F}_{\rm reg} \subset \mathcal{F}$. In subsection 2.1 we prove Theorem 2. In subsection 2.3-2.5 we prove a part of the claim of Theorem 4: the estimate for the codimension of the complement $\mathcal{F} \setminus \mathcal{F}_{\rm reg}$; in subsection 2.5 we also consider some immediate geometric implications of the conditions of general position.

2.1. The double spaces of general position. The open subset $\mathcal{W}_{\rm reg} \subset \mathcal{W}$ of hypersurfaces of degree $2M$ in $\mathbb{P} = \mathbb{P}^M$ is defined by local conditions, which a hypersurface $W \in \mathcal{W}_{\rm reg}$ must satisfy at *every* point $o \in W$. These conditions depend on whether the point $o \in W$ is non-singular or singular.

First, let us consider the condition of general position for a **non-singular point** $o \in W$. Let $(z_1, \dots, z_M)$ be a system of affine coordinates with the origin at the point o and

$$w = q_1 + q_2 + \ldots + q_{2M}$$

the affine equation of the branch hypersurface W, where the polynomials $q_i(z_*)$ are homogeneous of degree $i = 1, \dots, 2M$. At a non-singular point $o \in W$ (that is, $q_1 \not\equiv 0$) the hypersurface W must satisfy the condition

(W1) the rank of the quadratic form $q_2|_{\{q_1=0\}}$ is at least 2.

Proposition 2.1. *Violation of the condition (W1) imposes on the coefficients of the quadratic form q_2 (with the linear form q_1 fixed)*

$$\frac{(M-2)(M-1)}{2}$$

independent conditions.

Proof is obvious. Q.E.D.

Now let us consider the condition of general position for a **singular point** $o \in W$. Let

$$w = q_2 + q_3 + \ldots + q_{2M}$$

be the affine equation of the branch hypersurface W with respect to a system of affine coordinates $(z_1, \dots, z_M)$ with the origin at the point o. At a singular point o the hypersurface W must satisfy the condition

(W2) the rank of the quadratic form q_2 is at least 4.

Proposition 2.2. *Violation of the condition (W2) imposes on the coefficients of the quadratic form q_2*

$$\frac{(M-2)(M-1)}{2}$$

independent conditions.

Proof is obvious Q.E.D.

Now we define the subset $\mathcal{W}_{\rm reg} \subset \mathcal{W}$, requiring that $W \in \mathcal{W}_{\rm reg}$ satisfies the condition (W1) at every non-singular and the condition (W2) at every singular point. Obviously, $\mathcal{W}_{\rm reg} \subset \mathcal{W}$ is a Zariski open subset (possibly, empty).

Proposition 2.3. *The following estimate holds:*

$$\operatorname{codim}((\mathcal{W}\setminus\mathcal{W}_{\rm reg})\subset\mathcal{W})\geqslant\frac{(M-4)(M-1)}{2}.$$

Proof is obtained by the standard arguments, see [31, Chapter 3]: one considers the incidence subvariety

$$\mathcal{I}=\{(o,W)\,|\,o\in W\}\subset\mathbb{P}\times\mathcal{W};$$

for a fixed point $o\in\mathbb{P}$ the codimension of the set of hypersurfaces $\mathcal{W}_{\rm non-reg}(o)$, containing that point and non-regular in it, is given by Propositions 2.1 and 2.2 (in the singular case M more independent conditions are added as $q_1\equiv 0$). After that one computes the dimension of the set

$$\mathcal{I}_{\rm non-reg}=\bigcup_{o\in\mathbb{P}}\{o\}\times\mathcal{W}_{\rm non-reg}(o)$$

and considers the projection onto $\mathcal{W}$. This completes the proof. Q.E.D.

Obviously, for any hypersurface $W\in\mathcal{W}_{\rm reg}$ the double cover $F\to\mathbb{P}$, branched over W, is an irreducible algebraic variety. Moreover, by the condition (W2) the variety F belongs to the class of varieties with quadratic singularities of rank at least 5 [5]. Recall that a variety $\mathcal{X}$ is a variety with quadratic singularities of rank at least r, if in a neighborhood of every point $o\in\mathcal{X}$ the variety $\mathcal{X}$ can be realized as a hypersurface in a non-singular variety $\mathcal{Y}$, and the local equation $\mathcal{X}$ at the point o is of the form $\beta_1(u_*)+\beta_2(u_*)+\ldots=0$, where (u_*) is a system of local parameters at the point $o\in\mathcal{Y}$, and either $\beta_1\not\equiv 0$, or $\beta_1\equiv 0$ and $\operatorname{rk}\beta_2\geqslant r$. It is clear that $\operatorname{codim}(\operatorname{Sing}\mathcal{X}\subset\mathcal{X})\geqslant r-1$, so that the variety F is factorial [1].

Furthermore, it is easy to show (see [5]), that the class of quadratic singularities of rank at least r is stable with respect to blow ups in the following sense. Let $B\subset\mathcal{X}$ be an irreducible subvariety. Then there exists an open set $\mathcal{U}\subset\mathcal{Y}$, such that $\mathcal{U}\cap B\neq\emptyset$, $\mathcal{U}\cap B$ is a non-singular algebraic variety and for its blow up

$$\sigma_B\colon\mathcal{U}^+\to\mathcal{U}$$

we have that $(\mathcal{X}\cap\mathcal{U})^+\subset\mathcal{U}^+$ is a variety of quadratic singularities of rank at least r. In order to see this, note the following simple fact: if $\mathcal{Z}\ni o$ is a non-singular divisor on $\mathcal{Y}$, where $\mathcal{Z}\neq\mathcal{X}$ and the scheme-theoretic restriction $\mathcal{X}|_{\mathcal{Z}}$ has at the point o a quadratic singularity of rank l, then $\mathcal{X}$ has at the point o a quadratic singularity of rank at least l. Now if $B\not\subset\operatorname{Sing}\mathcal{X}$, then the claim about stability is obvious. Therefore, we may assume that $B\subset\operatorname{Sing}\mathcal{X}$. The open set $\mathcal{U}\subset\mathcal{Y}$ can be chosen in such a way that $B\cap\mathcal{U}$ is a non-singular subvariety and the rank of quadratic points $o\in B\cap\mathcal{U}$ is constant and equal to $l\geqslant r$. But then in the exceptional divisor $\mathcal{E}=\sigma_B^{-1}(B\cap\mathcal{U})$ the divisor $(\mathcal{X}\cap\mathcal{U})^+\cap\mathcal{E}$ is a fibration into quadrics of rank l, so that $(\mathcal{X}\cap\mathcal{U})^+\cap\mathcal{E}$ has at most quadratic singularities of rank at least l. Therefore, $(\mathcal{X}\cap\mathcal{U})^+\subset\mathcal{U}^+$ has quadratic singularities of rank at least r as well, according to the remark above. For an explicit analytic proof, see [5].

The stability with respect to blow ups implies that the singularities of the variety F are terminal (for the particular case of one blow up it is obvious: the discrepancy of an irreducible exceptional divisor $(\mathcal{X}\cap\mathcal{U})^+\cap\mathcal{E}$ with respect to $\mathcal{X}$ is positive; every exceptional divisor over $\mathcal{X}$ can be realized by a sequence of blow ups of the centres). Finally, F satisfies the condition (ii) of Theorem 1, that is, the condition of divisorial canonicity, see the proof of part (ii) of Theorem 2 in [25] and Theorem 4 in [30]. This completes the proof of Theorem 2. Q.E.D.

2.2. Fano hypersurfaces of general position. As in the case of double space, the open subset $\mathcal{F}_{\rm reg}\subset\mathcal{F}$ of hypersurfaces of degree M in $\mathbb{P}=\mathbb{P}^M$ is defined by the local conditions, which a hypersurface $F\in\mathcal{F}_{\rm reg}$ should satisfy at every point $o\in F$. Again these conditions are different for non-singular and singular points $o\in F$. Consider first the conditions of general position for a **non-singular point** $o\in F$.

Let $(z_1,\dots,z_M)$ be a system of affine coordinates with the origin at the point o and

$$w=q_1+q_2+q_3+\dots+q_M$$

the affine equation of the hypersurface F, where the polynomials $q_i(z_*)$ are homogeneous of degree $i=1,\dots,M$. Here is the list of conditions of general position, which a hypersurface F should satisfy at a non-singular point o.

(R1.1) The sequence

$$q_1,q_2,\dots,q_{M-1}$$

is regular in the local ring $\mathcal{O}_{o,\mathbb{P}}$, that is, the system of equations

$$q_1=q_2=\dots=q_{M-1}=0$$

defines a one-dimensional subset, a finite set of lines in $\mathbb{P}$, passing through the point o. In particular, $q_1\not\equiv 0$.

The equation $q_1=0$ defines the tangent space T_oF (which we, depending on what we need, will consider either a linear subspace in $\mathbb{C}^M$, or as its closure, a hyperplane in $\mathbb{P}$). Now set $\bar{q}_i=q_i|_{\{q_1=0\}}$ for $i=2,\dots,M$: these are polynomials on the linear space $T_oF\cong\mathbb{C}^{M-1}$. The condition (R1.1) means the regularity of the sequence

$$\bar{q}_2,\bar{q}_3,\dots,\bar{q}_{M-1}.$$

Such form is more convenient for estimating the codimension of the set of hypersurfaces which do not satisfy the regularity condition.

(R1.2) The quadratic form $\bar{q}_2$ on the space T_oF is of rank at least 6, and the linear span of every irreducible component of the closed algebraic set

$$\{q_1=q_2=q_3=0\}$$

in $\mathbb{C}^M$ is the hyperplane $\{q_1=0\}$, that is, the tangent hyperplane T_oF.

An equivalent wording of this condition: every irreducible component of the closed set $\{\bar{q}_2=\bar{q}_3=0\}$ in $\mathbb{P}^{M-2}=\mathbb{P}(\{q_1=0\})$ is non-degenerate.

(R1.3) For any hyperplane $P \subset \mathbb{P}$, $P \ni o$, different from the tangent hyperplane $T_oF \subset \mathbb{P}$, the algebraic cycle of scheme-theoretic intersection of hyperplanes P, T_oF, the projective quadric $\overline{\{q_2 = 0\}} \subset \mathbb{P}$ and F, that is, the cycle,

$$(P \circ \overline{\{q_1 = 0\}} \circ \overline{\{q_2 = 0\}} \circ F),$$

is irreducible and reduced. (The line above means the closure in $\mathbb{P}$ and the operation $\circ$ of taking the cycle of scheme-theoretic intersection is considered here on the space $\mathbb{P}$, too.)

Now let us consider the conditions of general position for a **singular point** $o \in F$. Let $(z_1, \dots, z_M)$ be a system of affine coordinates with the origin at the point o and

$$f = q_2 + q_3 + \dots + q_M$$

the affine equation of the hypersurface F, where the polynomials $q_i(z_*)$ are homogeneous of degree $i = 2, \dots, M$. Let us list the conditions of general position which must be satisfied for the hypersurface F at a singular point o.

(R2.1) For any linear subspace $\Pi \subset \mathbb{C}^M$ of codimension $c \in \{0, 1, 2\}$ the sequence

$$q_2|_\Pi, \dots, q_{M-c}|_\Pi \tag{5}$$

is regular in the ring $\mathcal{O}_{o,\Pi}$, that is, the system of equations

$$q_2|_{\mathbb{P}(\Pi)} = \dots = q_{M-c}|_{\mathbb{P}(\Pi)} = 0$$

defines in the space $\mathbb{P}(\Pi) \cong \mathbb{P}^{M-c-1}$ a finite set of points.

(R2.2) The quadratic form $q_2(z_*)$ is of rank at least 8.

(R2.3) Now let us consider $(z_1, \dots, z_M)$ as homogeneous coordinates $(z_1 : \dots : z_M)$ on $\mathbb{P}^{M-1}$. The divisor

$$\{q_3|_{\{q_2=0\}} = 0\}$$

on the quadric $\{q_2 = 0\}$ is not a sum of three (not necessarily distinct) hyperplane sections of this quadric, taken from the same linear pencil.

Now arguing in the word for word the same way as in subsection . 2.1, we conclude that any hypersurface $F \in \mathcal{F}_{\text{reg}}$ is an irreducible projective variety with factorial terminal singularities. Obviously, $K_F = -H_F$ and $\text{Pic}\, F = \mathbb{Z}H_F$, where H_F is the class of a hyperplane section $F \subset \mathbb{P}$, that is, F is a Fano variety of index one. In order to prove Theorem 4, we have to show the following two facts:

— the inequality (2),

— the divisorial log-canonicity of the hypersurface $F \in \mathcal{F}_{\text{reg}}$, that is, the condition (ii) of Theorem 1 for the variety F.

These two tasks are dealt with in the remaining part of this section and §3, respectively.

2.3. The conditions of general position at a non-singular point. Let $o \in F$ be a non-singular point. Fix an arbitrary non-zero linear form q_1 and consider the affine space of polynomials

$$q_1 + \mathcal{P}^{\mathrm{sing}} = \{q_1 + q_2 + \ldots + q_M\},$$

where $\mathcal{P}^{\mathrm{sing}}$ is the space of polynomials of the form $f = q_2 + q_3 + \ldots + q_M$. Let $\mathcal{P}_i \subset \{q_1 + \mathcal{P}^{\mathrm{sing}}\}$, $i = 1, 2, 3$, be the closures of the subsets, consisting of such polynomials f, which do not satisfy the condition (R1.i), respectively. Set

$$c_i = \mathrm{codim}(\mathcal{P}_i \subset \{q_1 + \mathcal{P}^{\mathrm{sing}}\}).$$

Proposition 2.4. *For $M \geqslant 8$ the following equality holds:*

$$\min\{c_1, c_2, c_3\} = c_2 = \frac{(M-6)(M-5)}{2}.$$

Proof is easy to obtain by elementary methods. First of all, by Lemma 2.1, shown below (where one must replace M by $(M-1)$), we obtain

$$c_1 = \frac{(M-1)(M-2)}{2} + 2.$$

Furthermore, a violation of the condition $\mathrm{rk}\,\bar{q}_2 \geqslant 6$ imposes on the coefficients of the quadratic form q_2

$$\frac{(M-6)(M-5)}{2} < c_1$$

independent conditions. Assuming the condition $\mathrm{rk}\,\bar{q}_2 \geqslant 6$ to be satisfied, we obtain that the quadric $\{\bar{q}_2 = 0\}$ is factorial. It is easy to check that reducibility or non-reducedness of the divisor $q_3|_{\{\bar{q}_2=0\}}$ on this quadric gives

$$\frac{M^3 - 6M^2 - 7M + 54}{6} > \frac{(M-6)(M-5)}{2}$$

independent conditions on the coefficients of the cubic form q_3.

Finally, let us consider a hyperplane $P \neq T_oF$ and the quadratic hypersurface

$$q_2|_{P \cap \{q_1=0\}} = 0.$$

Its rank is at least 5, so it is still factorial. Let us estimate from below the number of independent conditions, which are imposed on the coefficients of the polynomials $q_3, \ldots, q_M$ if the condition (R1.3) is violated. Define the values $v(\mu)$, $\mu = 0, 1, 2, 3$, by the table

μ	0	1	2	3
$v(\mu)$	0	1	M	$\frac{1}{2}M(M+1)-1$

and set

$$f(j,\mu)=\binom{j+M-1}{M-1}-\binom{j+M-3}{M-1}-v(\mu)+v(\max(0,\mu-2)).$$

Now, using the factoriality of the quadric, we obtain the estimate

$$c_3\geqslant f(M,3)-(M-2)-$$

$$-\max\left[\max_{M-1\geqslant j\geqslant 2}(f(j,2)+f(M-j,1)),\max_{M-1\geqslant j\geqslant 3}(f(j,3)+f(M-j,0))\right].$$

An elementary check shows that the minimum of the right hand side is strictly higher than c_2 (and for $M\to\infty$ grows exponentially). Q.E.D. for the proposition.

2.4. The conditions of general position at a singular point. Recall that $\mathcal{P}^{\mathrm{sing}}$ is the space of polynomials of the form

$$f=q_2+q_3+\ldots+q_M$$

in the variables $z_*=(z_1,\ldots,z_M)$, where $q_i(z_*)$ are homogeneous of degree i. Let $\mathcal{P}^{\mathrm{sing}}_{\mathrm{reg}}\subset\mathcal{P}^{\mathrm{sing}}$ be the subset of polynomials satisfying the conditions (R2.1-R2.3).

Proposition 2.5. *The following estimate holds:*

$$\mathrm{codim}(\overline{(\mathcal{P}^{\mathrm{sing}}\backslash\mathcal{P}^{\mathrm{sing}}_{\mathrm{reg}})}\subset\mathcal{P}^{\mathrm{sing}})=\frac{(M-7)(M-6)}{2}.$$

Proof. It is sufficient to show that violation of each of the conditions (R2.1-R2.3) at the point $o=(0,\ldots,0)$ separately imposes on the polynomial f at least $(M-7)(M-6)/2$ independent conditions. It is easy to check that violation of the condition (R2.2) imposes on the coefficients of the quadratic form $q_2(z_*)$ precisely $(M-7)(M-6)/2$ independent conditions. Therefore, considering the condition (R2.3), we may assume that the condition (R2.2) is satisfied; in particular, the quadric $\{q_2=0\}$) is factorial and violation of the condition (R2.3) imposes on the coefficients of the cubic form $q_3(z_*)$ (with the polynomial q_2 fixed)

$$M\frac{M^2+3M-16}{6}\geqslant\frac{(M-7)(M-6)}{2}$$

independent conditions for $M\geqslant 4$. It remains to consider the case when the condition (R2.1) is violated.

Lemma 2.1. *Violation of the condition (R2.1) for one value of the parameter* $c=0$ *imposes on the coefficients of the polynomial* f

$$\frac{M(M-1)}{2}+2 \tag{6}$$

independent conditions.

Proof is obtained by the standard methods [31, Chapter 3]. We just remind the scheme of arguments. Fix the first moment when the sequence of polynomials $q_2, \ldots, q_M$ becomes non-regular: assume that the regularity is first violated for q_k, that is, the closed set $\{q_2 = \ldots = q_{k-1} = 0\}$ has the "correct" codimension $k-2$ and q_k vanishes on one of the components of that set. For $k \leqslant M-1$ we apply the method of [20] and obtain that violation of the regularity condition imposes on the coefficients of the polynomial f at least

$$\binom{M+1}{k} \geqslant \frac{(M+1)M}{2}$$

independent conditions; the right hand side of the last inequality is strictly higher than (6), which is what we need.

Let us consider the last option:

$$\{q_2 = \ldots = q_{M-1} = 0\} \subset \mathbb{P}^{M-1}$$

is a one-dimensional closed set and q_M vanishes on one of its irreducible components, say B. The case when $B \subset \mathbb{P}^{M-1}$ is a line is a special one: it is easy to check that vanishing on a line in $\mathbb{P}^{M-1}$ imposes on the polynomials $q_2, \ldots, q_M$ in total precisely (6) independent conditions. Therefore, we may assume that B is not a line, that is, $\dim B < \langle B \rangle = k \geqslant 2$. Now we apply the method suggested in [22], fixing k and the linear subspace $\langle B \rangle$. To begin with, consider the case $k \leqslant M-2$. In that case there are indices

$$i_1, \ldots, i_{k-1} \in \{2, \ldots, M-1\},$$

such that the restrictions $q_{i_1}|_{\langle B \rangle}, \ldots, q_{i_{k-1}}|_{\langle B \rangle}$ form a good sequence and B is one of its associated subvarieties (see [22, Sec.3, Proposition 4], the details of this procedure are described in the proof of the cited proposition). Taking into account that $B \subset \langle B \rangle$ is by construction a non-degenerate curve, we see that decomposable polynomials of the form $l_1 \ldots l_a$, where l_i are linear forms on $\langle B \rangle \cong \mathbb{P}^k$, can not vanish on B. This gives $jk+1$ independent conditions for each of the polynomials q_j for $j \notin \{i_1, \ldots, i_{k-1}\}$, so that in total we get at least

$$\frac{k(M-k)(M-k+1)}{2} + M - 2k - 1$$

independent conditions for these polynomials (the minimum is attained for $i_1 = M-k+1, \ldots, i_{k-1} = M-1$). Taking into account the condition $q_M|_B \equiv 0$ and the dimension of the Grassmanian of k-dimensional subspaces in $\mathbb{P}^{M-1}$, we obtain at least

$$M^2 - kM + k^2 - M + k + 1$$

independent conditions for f. It is easy to check that the last number is not smaller than (6).

Finally, if $k = M-1$, that is, B is a non-degenerate curve in $\mathbb{P}^{M-1}$, then the condition $q_M|_B \equiv 0$ gives at least $M(M-1)+1$ independent conditions for q_M. Proof of Lemma 2.1 is complete. Q.E.D.

Now let us complete the proof of Proposition 2.5.

For a *fixed* linear subspace $\Pi\subset\mathbb{C}^M$ of codimension $c\in\{0,1,2\}$ violation of regularity of the sequence (5) imposes on the polynomial f at least $(M-c)(M-c-1)/2+2$ independent conditions. Subtracting the dimension of the Grassmanian of subspaces of codimension c in $\mathbb{C}^M$, we get the least value $(M-3)(M-6)/2$ for $c=2$. This completes the proof of Proposition 2.5. Q.E.D.

2.5. Estimating the codimension of the complement to the set $\mathcal{F}_{\rm reg}$. Recall that $F\in\mathcal{F}_{\rm reg}$ if and only if at every non-singular point $o\in F$ the conditions (R1.1-3) are satisfied, and at every singular point $o\in F$ the conditions (R2.1-3) are satisfied. Propositions 2.4 and 2.5 imply the following fact.

Proposition 2.6. *The following estimate holds:*

$$\mathrm{codim}((\mathcal{F}\setminus\mathcal{F}_{\rm reg})\subset\mathcal{F})\geqslant\frac{(M-7)(M-6)}{2}-5.$$

Proof is completely similar to the proof of Proposition 2.3 and follows from Propositions 2.4 and 2.5.

Now let us consider some geometric facts which follow immediately from the conditions of general position. These facts will be needed in §3 to exclude log maximal singularities. In [25] it was shown that for any effective divisor $D\sim nH$ on F (where we write H in stead of H_F to simplify the notations) the pair $(F,\frac{1}{n}D)$ is canonical at non-singular points $o\in F$. This fact will be used without special references. Now let $D_2=\{q_2|_F=0\}$ be the first hypertangent divisor, so that we have $D_2^+\in|2H-3E|$. Recall that $E\subset\mathbb{P}^{M-1}$ is an irreducible quadric of rank at least 8. Obviously, the divisor $D_2\in|2H|$ satisfies the equality

$$\frac{\mathrm{mult}_o}{\deg}D_2=\frac{3}{M}.$$

Here and below the symbol $\mathrm{mult}_o/\deg$ means the ratio of multiplicity at the point o to the degree.

Lemma 2.2. *Let $P\subset F$ be the section of the hypersurface F by an arbitrary linear subspace in $\mathbb{P}$ of codimension two, containing the point o. Then the restriction $D_2|_P$ is an irreducible reduced divisor on the hypersurface $P\subset\mathbb{P}^{M-2}$.*

Proof. The variety P has at most quadratic singularities of rank at least 6 and for that reason it is factorial. Therefore, reducibility or non-reducedness of the divisor $D_2|_P$ means, that the equality $D_2|_P=H_1+H_2$ holds, where H_i are possibly coinciding hyperplane sections of P. By the condition (R2.2) the equalities $\mathrm{mult}_o H_i=2$ hold. However, $\mathrm{mult}_o D_2|_P=6$. Therefore, $D_2|_P$ can not break into two hyperplane sections. Q.E.D. for the lemma.

Proposition 2.7. *The pair $(F,\frac{1}{2}D_2)$ has no non log canonical singularities, the centre of which on F contains the point o: $LCS(F,\frac{1}{2}D_2)\not\ni o$.*

Proof. Assume the converse. In any case

$$\mathrm{codim}(LCS\left(F,\frac{1}{2}D_2\right)\subset F)\geqslant 6,$$

so that consider the section $P \subset F$ of the hypersurface F by a generic linear subspace of dimension 5, containing the point o. Then the pair $(P, \frac{1}{2}D_2|_P)$ has the point o as an isolated centre of a non log canonical singularity. Let $\sigma_P\colon P^+ \to P$ be the blow up of the non-degenerate quadratic singularity $o \in P$ so that $E_P = E \cap P^+$ is a non-singular exceptional quadric in $\mathbb{P}^4$. Since

$$\frac{1}{2}(D_2|_P)^+ \sim H_P - \frac{3}{2}E_P \quad \text{and} \quad a(E_P, P) = 2 > \frac{3}{2}$$

(where H_P is the class of a hyperplane section of $P \subset \mathbb{P}^5$), the pair $(P^+, \frac{1}{2}(D_2|_P)^+)$ is not log canonical. The union $LCS(P^+, \frac{1}{2}(D_2|_P)^+)$ of all centres of non log canonical singularities of that pair, intersecting E_P, is a connected closed subset of the exceptional quadric E_P, every irreducible component S_P of which satisfies the inequality $\operatorname{mult}_{S_P}(D_2|_P)^+ \geqslant 3$. Coming back to the original pair $(F, \frac{1}{2}D_2)$, we see that for some irreducible subvariety $S \subset E$ the inequality $\operatorname{mult}_S D_2^+ \geqslant 3$ holds, where $S \cap P^+ = S_P$, so that $\operatorname{codim}(S \subset E) \in \{1, 2, 3\}$.

However, the case $\operatorname{codim}(S \subset E) = 3$ is impossible: by the connectedness principle this equality means that S_P is a point, and then $S \subset E$ is a linear subspace of codimension 3, which is impossible if $\operatorname{rk} q_2 \geqslant 8$ (a 7-dimensional non-singular quadric does not contain linear subspaces of codimension 3).

Consider the case $\operatorname{codim}(S \subset E) = 2$. Let $\Pi \subset E$ be a general linear subspace of maximal dimension. Then $D_2^+|_\Pi$ is a cubic hypersurface that has multiplicity 3 along an irreducible subvariety $S_\Pi = S \cap \Pi$ of codimension 2. Therefore, $D_2^+|_\Pi$ is a sum of three (not necessarily distinct) hyperplanes in Π, containing the linear subspace $S_\Pi \subset \Pi$ of codimension 2, and for that reason $D_2^+|_E$ is a sum of three (not necessarily distinct) hyperplane sections from the same linear pencil as well, and S is the intersection of the quadric E and a linear subspace of codimension 2. However, this is impossible by the condition (R2.3).

Finally, if $\operatorname{codim}(S \subset E) = 1$, then $D_2^+|_E = 3S$ is a triple hyperplane section of the quadric E, which is impossible by the condition (R2.3).

This completes the proof of Proposition 2.7. Q.E.D.

Here is one more fact that will be useful later.

Proposition 2.8. *For any hyperplane section $\Delta \ni o$ of the hypersurface F the pair (F, Δ) is log canonical.*

Proof. This follows from a well known fact (see, for instance, [3, 30]): if $(p \in X)$ is a germ of a non-degenerate quadratic three-dimensional singularity, $\sigma\colon \widetilde{X} \to X$ its resolution with the exceptional quadric $E_X \cong \mathbb{P}^1 \times \mathbb{P}^1$ and D_X a germ of an effective divisor such that $o \in D_X$ and $\widetilde{D}_X \sim -\beta E_X$, then the pair $(X, \frac{1}{\beta}D_X)$ is log canonical at the point o. Q.E.D.

3 Exclusion of maximal singularities

In this section we complete the proof of Theorem 4. The symbol F stands for a fixed hypersurface of degree M in $\mathbb{P}$, satisfying the regularity conditions: $F \in \mathcal{F}_{\text{reg}}$.

As we mentioned in §2, in [25] it was shown that the pair $(F,\frac{1}{n}D)$ has no maximal singularities, the centre of which is not contained in the closed set $\mathrm{Sing}\, F$, for every effective divisor $D\sim nH$. In [5] it was shown that for any mobile linear system $\Sigma\subset|nH|$ the pair $(F,\frac{1}{n}D)$ is canonical for a general divisor $D\in\Sigma$, that is, Σ has no maximal singularities. Therefore, in order to complete the proof of Theorem 4 it is sufficient to show that for any effective divisor $D\sim nH$ the pair $(F,\frac{1}{n}D)$ is log canonical, and we may assume only those log maximal singularities, the centre of which is contained in $\mathrm{Sing}\, F$.

In subsection 3.1 we carry out preparatory work: by means of the technique of hypertangent divisors we obtain estimates for the ratio $\mathrm{mult}_o/\deg$ for certain classes of irreducible subvarieties of the hypersurface F. After that we fix a pair $(F,\frac{1}{n}D)$ and assume that it is not log canonical. The aim is to bring this assumption to a contradiction. Let $B^*\subset\mathrm{Sing}\, F$ be the centre of the log maximal singularity of the divisor D, $o\in B^*$ a point of general position, $F^+\to F$ its blow up, D^+ the strict transform of the divisor D. In subsection 3.2 we study the properties of the pair $(F^+,\frac{1}{n}D^+)$: we show that this pair has a non log canonical singularity, the centre of which is a subvariety of the exceptional divisor of the blow up of the point o. After that in subsections 3.2 and 3.3 we show that this is impossible, which completes the proof of Theorem 4.

3.1. The method of hypertangent divisors. Fix a singular point $o\in F$, a system of coordinates $(z_1,\dots,z_M)$ on $\mathbb{P}$ with the origin at that point and the equation $f=q_2+\dots q_M$ of the hypersurface F.

Proposition 3.1. *Assume that the variety F satisfies the conditions (R2.1, R2.2) at the singular point o. Then the following claims hold.*

(i) *For every irreducible subvariety of codimension 2 $Y\subset F$ the following inequality holds:*

$$\frac{\mathrm{mult}_o}{\deg}Y\leqslant\frac{4}{M}.$$

(ii) *Let $\Delta\ni o$ be an arbitrary hyperplane section of the hypersurface F. For every prime divisor $Y\subset\Delta$ the following inequality holds:*

$$\frac{\mathrm{mult}_o}{\deg}Y\leqslant\frac{3}{M}.$$

(iii) *Let $P\ni o$ be the section of the hypersurface F by an arbitrary linear subspace of codimension two. For every prime divisor $Y\subset P$ the following inequality holds:*

$$\frac{\mathrm{mult}_o}{\deg}Y\leqslant\frac{4}{M}.$$

Proof is obtained by means of the method of hypertangent divisors [31, Chapter 3]. For $k=2,\dots,M-1$ let

$$\Lambda_k=\left|\sum_{i=2}^{k}s_{k-i}(q_2+\dots+q_i)|_F=0\right|$$

be the *k-th hypertangent linear system*, where $s_j(z_*)$ are all possible homogeneous polynomials of degree j. For the blow up $\sigma\colon F^+ \to F$ of the point o with the exceptional divisor $E=\sigma^{-1}(o)$, naturally realized as a quadric in $\mathbb{P}^{M-1}$, we have

$$\Lambda_k^+ \subset |kH-(k+1)E|$$

(where Λ_k^+ is the strict transform of the system Λ_k on F^+). Let $D_k \in \Lambda_k$, $k=2,\dots,M-1$ be general hypertangent divisors.

Let us show the claim (i). By the condition (R2.1) the equality

$$\mathrm{codim}_o(\mathrm{Bs}\,\Lambda_k \subset F)=k-1 \tag{7}$$

holds, where the symbol codim_o means the codimension in a neighborhood of the point o; therefore,

$$Y\cap D_4\cap D_5\cap\ldots\cap D_{M-1}$$

in a neighborhood of the point o is a closed one-dimensional set. We construct a sequence of irreducible subvarieties $Y_i\subset F$ of codimension i: $Y_2=Y$ and Y_{i+1} is an irreducible component of the effective cycle $(Y_i\circ D_{i+2})$ with the maximal value of the ratio $\mathrm{mult}_o/\deg$. The cycle $(Y_i\circ D_{i+2})$ every time is well defined, because by the inequality (7) we have $Y_i\not\subset D_{i+2}$ for a general hypertangent divisor D_{i+2}. By the construction of hypertangent linear system, at every step of our procedure the inequality

$$\frac{\mathrm{mult}_o}{\deg}Y_{i+1}\geqslant\frac{i+3}{i+2}\cdot\frac{\mathrm{mult}_o}{\deg}Y_i$$

holds, so that for the curve Y_{M-2} we have the estimates

$$1\geqslant\frac{\mathrm{mult}_o}{\deg}Y_{M-2}\geqslant\frac{\mathrm{mult}_o}{\deg}Y\cdot\frac{5}{4}\cdot\frac{6}{5}\cdot\ldots\cdot\frac{M}{M-1},$$

which implies the claim (i).

Let us prove the claim (ii). By Lemma 2.2, the divisor $D_2|_\Delta$ is irreducible and reduced, and by the condition (R2.1) it satisfies the equality

$$\frac{\mathrm{mult}_o}{\deg}D_2|_\Delta=\frac{3}{M}.$$

Therefore, we may assume that $Y\neq D_2|_\Delta$, so that $Y\not\subset D_2$ and the effective cycle of codimension two $(Y\circ D_2)$ on Δ is well defined and satisfies the inequality

$$\frac{\mathrm{mult}_o}{\deg}(Y\circ D_2)\geqslant\frac{3}{2}\cdot\frac{\mathrm{mult}_o}{\deg}Y.$$

Let Y_2 be an irreducible component of that cycle with the maximal value of the ratio $\mathrm{mult}_o/\deg$. Applying to Y_2 the technique of hypertangent divisors in precisely the same way as in the part (i) above, we see that by the condition (R2.1) the intersection

$$Y_2\cap D_4|_\Delta\cap D_5|_\Delta\cap\ldots\cap D_{M-2}|_\Delta$$

in a neighborhood of the point o is a one-dimensional closed set, where $D_4 \in \Lambda_4, \ldots,$ $D_{M-2} \in \Lambda_{M-2}$ are general hypertangent divisors (note that the last hypertangent divisor is D_{M-2}, and not D_{M-1}, as in the part (i), because the dimension of Δ is one less than the dimension of F and the condition (R2.1) provides the regularity of the truncated sequence $q_2|_\Pi, \ldots, q_{M-1}|_\Pi$, where in this case Π is a hyperplane, cutting out Δ on F). Now, arguing in the word for word same way as in the proof of the claim (i), we obtain the estimate

$$1 \geqslant \frac{\mathrm{mult}_o}{\deg} Y \cdot \frac{3}{2} \cdot \frac{5}{4} \cdot \frac{6}{5} \cdot \ldots \cdot \frac{M-1}{M-2},$$

which implies that

$$\frac{\mathrm{mult}_o}{\deg} Y \leqslant \frac{8}{3(M-1)}.$$

For $M \geqslant 9$ the right hand side of the inequality does not exceed $3/M$, which proves the claim (ii).

Let us show the claim (iii). We argue in the word for word same way as in the proof of the part (ii), with the only difference: in order to estimate the multiplicity of the cycle $(Y \circ D_2|_P)$ at the point o we use the hypertangent divisors

$$D_4|_P, D_5|_P, \ldots, D_{M-3}|_P$$

(one less than above), so that we get the estimate

$$1 \geqslant \frac{\mathrm{mult}_o}{\deg} Y \cdot \frac{3}{2} \cdot \frac{5}{4} \cdot \frac{6}{5} \cdot \ldots \cdot \frac{M-2}{M-3},$$

which implies that

$$\frac{\mathrm{mult}_o}{\deg} Y \leqslant \frac{8}{3(M-2)}.$$

For $M \geqslant 6$ the right hand side of the inequality does not exceed $4/M$, which proves the claim (iii).

Proof of Proposition 3.1 is complete. Q.E.D.

Let us resume the proof of Theorem 4.

3.2. The blow up of a singular point. Assume that the pair $(F, \frac{1}{n}D)$ is not log canonical for some divisor $D \in |nH|$, that is, for some prime divisor E^* over F, that is, a prime divisor $E^* \subset \widetilde{F}$, where $\psi\colon \widetilde{F} \to F$ is some birational morphism, $\widetilde{F}$ is non-singular and projective, the *log Noether-Fano inequality* holds:

$$\mathrm{ord}_{E^*} \psi^* D > n(a(E^*) + 1).$$

By linearity of the inequality in D and n we may assume the divisor D to be prime. Let $B^* = \psi(E^*) \subset F$ be the centre of the log maximal singularity E^*. We known that $B^* \subset \mathrm{Sing}\, F$; in particular, $\mathrm{codim}(B^* \subset F) \geqslant 7$. Let $o \in B^*$ be a point of general position, $\varphi\colon F^+ \to F$ its blow up, $E \subset F^+$ the exceptional quadric.

Consider the first hypertangent divisor $D_2 \in |2H|$ at the point o. By Lemma 2.2, the divisor D_2 is irreducible and reduced, and by Proposition 2.7, the pair $(F, \frac{1}{2}D_2)$ is log canonical at the point o. Therefore, $D \neq D_2$.

Proposition 3.2. *The following inequality holds*

$$\mathrm{mult}_o D \leqslant \frac{8}{3}n.$$

Proof. Consider the effective cycle $(D \circ D_2)$ of codimension two. Obviously,

$$\frac{\mathrm{mult}_o}{\deg}(D \circ D_2) \geqslant \frac{3}{2} \cdot \frac{\mathrm{mult}_o}{\deg} D,$$

however, by Proposition 3.1, (i), the left hand part of this inequality does not exceed $(4/M)$. Since $\deg D = nM$, Proposition 3.2 is shown. Q.E.D.

Write down $D^+ \sim nH - \nu E$, where $\nu \leqslant \frac{4}{3}n$.

Let us consider the section P of the hypersurface F by a general 5-dimensional linear subspace, containing the point o. Let P^+ be the strict transform of P on F^+ and $E_P = P^+ \cap E$ a non-singular three-dimensional quadric. Set also $D_P = D|_P$. Obviously, the pair $(P, \frac{1}{n}D_P)$ has the point o as an isolated centre of a non log canonical singularity. Since $a(E_P) = 2$ and $D_P^+ \sim nH_P - \nu E_P$ (where H_P is the class of a hyperplane section of the variety P), where $\nu \leqslant \frac{4}{3}n < 2n$, the pair $(P^+, \frac{1}{n}D_P^+)$ is not log canonical and the union $LCS_E(P^+, \frac{1}{n}D_P^+)$ of centres of all non log canonical singularities of that pair, intersecting E_P, is a connected closed subset of the exceptional quadric E_P. Let S_P be an irreducible component of that set. Obviously, the inequality

$$\mathrm{mult}_{S_P} D_P^+ > n$$

holds. Furthermore, $\mathrm{codim}(S_P \subset E_P) \in \{1, 2, 3\}$. Returning to the original pair $(F, \frac{1}{n}D)$, we see that there is a non log canonical singularity of the pair $(F^+, \frac{1}{n}D^+)$, the centre of which is a subvariety $S \subset E$, such that $S \cap E_P = S_P$ and, in particular, $\mathrm{codim}(S \subset E) \in \{1, 2, 3\}$.

Note at once that the case $\mathrm{codim}(S \subset E) = 3$ is impossible: by the connectedness principle in that case S_P is a point and for that reason S is a linear subspace of codimension 3 on the quadric E of rank at least 8, which is impossible.

It is not hard to exclude the case $\mathrm{codim}(S \subset E) = 1$, either. Assume that it does take place. Then the divisor S is cut out on E by a hypersurface of degree $d_S \geqslant 1$. Let H_E be the class of a hyperplane section of the quadric E. The divisor $D^+|_E \sim \nu H_E$, so that

$$\frac{4}{3}n \geqslant \nu > nd_S$$

and for that reason S is a hyperplane section of the quadric E. Let $\Delta \in |H|$ be the uniquely determined hyperplane section of the hypersurface F, such that $\Delta \ni o$ and $\Delta^+ \cap E = S$. The pair (F^+, Δ^+) is log canonical and for that reason $D \neq \Delta$. For the effective cycle $(D \circ \Delta)$ of codimension two on F we have

$$\mathrm{mult}_o(D \circ \Delta) \geqslant 2\nu + 2\,\mathrm{mult}_S D^+ > 4n,$$

so that

$$\frac{\mathrm{mult}_o}{\deg}(D \circ \Delta) > \frac{4}{M},$$

which contradicts Proposition 3.1. This excludes the case of a divisorial centre.

3.3. The case of codimension two. Starting from this moment, assume that $\mathrm{codim}(S \subset E) = 2$.

Lemma 3.1. *The subvariety S is contained in some hyperplane section of the quadric E.*

Proof. Since $\mathrm{mult}_S D^+ > n$ and $D^+|_E \sim \nu H_E$ with $\nu \leqslant \frac{4}{3}n$, for every secant line $L \subset E$ of the subvariety S we have $L \subset D^+$. Let $\Pi \subset E$ be a linear space of maximal dimension and of general position and $S_\Pi = S \cap \Pi$. The secant lines of the closed set $S_\Pi \subset \Pi$ of codimension two can not sweep out Π, since $E \not\subset D^+$. Therefore, there are two options (see [30, Lemma 2.3]):

1) the secant lines of the set S_Π sweep out a hyperplane in Π,

2) $S_\Pi \subset \Pi$ is a linear subspace of codimension two.

In the first case the secant lines $L \subset E$ of the set S sweep out a divisor on E, which ca only be a hyerplane section of the quadric E. In the second case S contains all its secant lines and is a section of E by a linear subspace of codimension two. Q.E.D. for the lemma.

As we have just shown, one of the two options takes place: either there is a unique hyperplane section Λ of the quadric E, containing S (Case 1), or $S = E \cap \Theta$, where Θ is a linear subspace of codimension two (Case 2). Let us study them separately.

Assume that **Case 1** takes place. Then S is cut out on Λ by a hypersurface of degree $d_S \geqslant 2$. Set

$$\mu = \mathrm{mult}_S D^+ \quad \text{and} \quad \gamma = \mathrm{mult}_\Lambda D^+,$$

where $\mu > n$ and $\mu \leqslant 2\nu \leqslant \frac{8}{3}n$.

Lemma 3.2. *The following inequality holds:*

$$\gamma \geqslant \frac{2\mu - \nu}{3}.$$

Proof is easy to obtain in the same way as the short proof of Lemma 3.5 in [21, subsection 3.7]. Let $L \subset \Lambda$ be a general secant line of the set S. Consider the section P of the hypersurface F by a general 4-plane in $\mathbb{P}$, such that $P \ni o$ and $P^+ \cap E$ contains the line L. Obviously, $o \in P$ is a non-degenerate quadratic point and $E_P = P^+ \cap E \cong \mathbb{P}^1 \times \mathbb{P}^1$ is a non-singular quadric in $\mathbb{P}^3$. Set $D_P = D|_P$. Obviously, $\gamma = \mathrm{mult}_L D_P^+$.

Let $\sigma_L \colon P_L \to P^+$ be the blow up of the line L, $E_L = \sigma_L^{-1}(L)$ the exceptional divisor; since $\mathcal{N}_{L/P^+} \cong \mathcal{O} \oplus \mathcal{O}_L(-1)$, the exceptional surface E_L is a ruled surface of the type $\mathbb{F}_1$, so that

$$\mathrm{Pic}\, E_L = \mathbb{Z}s \oplus \mathbb{Z}f,$$

where s and f are the classes of the exceptional section and the fibre, respectively. Furthermore, $E_L|_{E_L} = -s - f$. Let D_L be the strict transform of D_P^+ on P_L. Obviously,

$$D_L \sim nH_P - \nu E_P - \gamma E_L$$

(where H_P is the class of a hyperplane section of P), so that

$$D_L|_{E_L} \sim \gamma s + (\gamma + \nu) f.$$

On the other hand, L is a general secant line of the set S and for that reason L contains at least two distinct points $p, q \in S$. Therefore, the divisor D_P^+ has at the points $p, q \in L$ the multiplicity μ and for that reason the effective 1-cycle $D_L|_{E_L}$ contains the corresponding fibres $\sigma_L^{-1}(p)$ and $\sigma_L^{-1}(q)$ over those points with multiplicity $(\mu - \gamma)$. Therefore,

$$\gamma + \nu \geqslant 2(\mu - \gamma),$$

whence follows the claim of the lemma. Q.E.D.

Now let us consider the uniquely determined hyperplane section Δ of the hypersurface $F \subset \mathbb{P}$, such that $\Delta \ni o$ and $\Delta^+ \cap E = \Lambda$. Set $D_\Delta = D|_\Delta$. Write down

$$D^+|_{\Delta^+} = D_\Delta^+ + a\Lambda.$$

Obviously,

$$\mathrm{mult}_o\, D_\Delta = 2(\nu + a) \geqslant 2\nu + 2\frac{2\mu - \nu}{3} = \frac{4}{3}(\mu + \nu) > \frac{8}{3}n.$$

Since as we noted above, the subvariety S is cut out on the quadric Λ by a hypersurface of degree $d_S \geqslant 2$, the divisor $D_\Delta^+ \sim nH_\Delta - (\nu + a)\Lambda$ can not contain S with multiplicity higher than

$$\frac{1}{d_S}(\nu + a) \leqslant \frac{\nu + a}{2}.$$

Since the pair $(F^+, \frac{1}{n}D^+)$ has a non log canonical singularity with the centre at S, the inversion of adjunction implies that the pair $\square = (\Delta^+, \frac{1}{n}(D_\Delta^+ + a\Lambda))$ has a non log canonical singularity with the centre at S as well. Recall that $\mathrm{codim}(S \subset \Delta^+) = 2$. Consider the blow up $\sigma_S\colon \widetilde{\Delta} \to \Delta^+$ of the subvariety S and denote by the symbol E_S the exceptional divisor $\sigma_S^{-1}(S)$. The following fact is well known.

Proposition 3.3. *For some irreducible divisor $S_1 \subset E_S$, such that the projection $\sigma_S|_{S_1}$ is birational, the inequality*

$$\mathrm{mult}_S(D_\Delta^+ + a\Lambda) + \mathrm{mult}_{S_1}(\widetilde{D}_\Delta + a\widetilde{\Lambda}) > 2n \qquad (8)$$

holds, where $\widetilde{D}_\Delta$ and $\widetilde{\Lambda}$ are the strict transforms of D_Δ^+ and Λ on $\widetilde{\Delta}$, respectively.

Proof: see Proposition 9 in [25].

Set $\mu_S = \mathrm{mult}_S\, D^+_\Delta$ and $\beta = \mathrm{mult}_{S_1}\, \widetilde{D}_\Delta$. Consider first the case of general position: $S_1 \neq E_S \cap \widetilde{\Lambda}$. In that case $S_1 \not\subset \widetilde{\Lambda}$ and the inequality (8) takes the following form:

$$\mu_S + \beta + a > 2n.$$

Since $\mu_S \geqslant \beta$, the more so $2\mu_S + a > 2n$. On the other hand, we noted above that $2\mu_S \leqslant \nu + a$. As a result, we obtain the estimate

$$\nu + 2a > 2n.$$

Therefore, $\mathrm{mult}_o\, D_\Delta > \nu + 2n > 3n$. However, $D_\Delta \sim nH_\Delta$ is an effective divisor on the hyperplane section Δ and by Proposition 3.1, (ii), it satisfies the inequality

$$\frac{\mathrm{mult}_o}{\deg} D_\Delta \leqslant \frac{3}{M}.$$

This contradiction excludes the case of general position. Therefore, we are left with the only option: $S_1 = E_S \cap \widetilde{\Lambda}$.

In that case the inequality (8) takes the following form:

$$\mu_S + \beta + 2a > 2n.$$

This inequality is weaker than the corresponding estimate in the case of general position, but as a compensation we obtain the additional inequality

$$2\mu_S + 2\beta \leqslant \nu + a$$

(the restriction $D^+_\Delta|_\Lambda$ is cut out by a hypersurface of degree $(\nu + a)$ and contains the divisor $S \sim d_S H_\Lambda$ with multiplicity at least $\mu_S + \beta$). Combining the last two estimates, we obtain the inequality

$$\nu + 5a > 4n,$$

which implies that $5(\nu + a) > 8n$ and so $\mathrm{mult}_o\, D_\Lambda > \frac{16}{5}n$; as we mentioned above, this contradicts Proposition 3.1, (ii). This completes the exclusion of Case 1.

Therefore, **Case 2** takes place: $S = E \cap \Theta$, where $\Theta \subset \mathbb{P}^{M-1}$ is a linear subspace of codimension two. Let $P \subset F$ be the section of the hypersurface F by the linear subspace of codimension two in $\mathbb{P}$, which is uniquely determined by the conditions $P \ni o$ and $P^+ \cap E = S$. Furthermore, let $|H - P|$ be the pencil of hyperplane sections of F, containing P. For a general hyperplane section $\Delta \in |H - P|$ we have:

- the divisor D does not contain Δ as a component, so that the effective cycle $(D \circ \Delta) = D_\Delta$ of codimension two on F is well defined; this cycle can be looked at as an effective divisor $D_\Delta \in |nH_\Delta|$ on the hypersurface $\Delta \subset \mathbb{P}^{M-1}$,

- for the strict transform D^+_Δ on F^+ the equality $\mathrm{mult}_S\, D^+_\Delta = \mathrm{mult}_S\, D^+$ holds.

Of course, the divisor D_Δ may contain P as a component. Write down $D_\Delta = G + aP$, where $a \in \mathbb{Z}_+$ and G is an effective divisor that does not contain P as a component, $G \in |(n-a)H_\Delta|$. Obviously, $G^+ \sim (n-a)H_\Delta - (\nu - a)E_\Delta$, where $E_\Delta = \Delta^+ \cap E$, and, besides,

$$\mathrm{mult}_S\, G^+ = \mathrm{mult}_S\, D^+ - a > n - a.$$

Set $m = n - a$. The effective cycle of codimension two $G_P = (F \circ P)$ on Δ is well defined and can be considered as an effective divisor $G_P \in |mH_P|$ on the hypersurface $P \subset \mathbb{P}^{M-2}$. The divisor G_P satisfies the inequality

$$\mathrm{mult}_o\, G_P \geqslant 2(\nu - a) + 2\, \mathrm{mult}_S\, G^+ > 4m.$$

This is impossible by Proposition 3.1, (iii).

Therefore, the assumption that the pair $(F, \frac{1}{n}D)$ is not log canonical for some divisor $D \sim nH$, leads to a contradiction.

Proof of Theorem 4 is complete.

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